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Note

Orthomodular semilattices

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Abstract

We present a simple equational characterization of (meet) semilattices with 0 where for each element p the interval $[0, p]$ is an orthomodular lattice or an ortholattice possibly satisfying the compatibility condition.

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By an *ortholattice* (see [1,6]) is meant a system $(L; \wedge, \vee, ', 0, 1)$ such that $(L; \wedge, \vee, 0, 1)$ is a bounded lattice and $'$ is a unary operation on L assigning to each $a \in L$ an element a' which is a complement of a in L, $a \leqslant b$ implies $b' \leqslant a'$ (antitony) and $a'' = a$ (involution). The element x' is called the *orthocomplement* of x. By an *orthomodular lattice* is meant an ortholattice satisfying the so-called *orthomodular law* (see [1,6]):

$$x \leqslant y \quad \text{implies } x \vee (x' \wedge y) = y$$

or, equivalently (see [1])

$$x \leqslant y \quad \text{implies } y \wedge (y' \vee x) = x.$$

Several authors tried to generalize the concept of an orthomodular lattice as follows: Janowitz [5] investigated the so-called generalized orthomodular lattices, i.e. lattices with 0 where for each element p the interval $[0, p]$ is an orthomodular lattice satisfying a certain compatibility condition (see below). Mayet-Ippolito [7] extended the concept of a generalized orthomodular lattice to a poset with 0, the so-called (weakly) generalized orthomodular poset. Hedlíková [4] introduced relatively orthomodular lattices which are lattices where every interval is an orthomodular lattice with a compatibility condition. However, for some investigations of quantum mechanic logic, it is a rather strong assumption to suppose that L is a lattice, see [1,3,6,7]; in fact, we work with semilattices only. It is the aim of this note to extend the aforementioned concepts of an ortholattice and an orthomodular lattice for (meet) semilattices. However, then we obtain in fact only a partial algebra since the operation join is defined only in every interval but not necessarily in the

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whole algebra. We are going to show how to avoid this discrepancy by using of a suitable binary operation introduced below.

In what follows, by a semilattice we shall mean a meet semilattice.

Lemma. *Let* $(L; \wedge)$ *be a semilattice and* $'$ *be an involutive unary operation on* L. *If* $'$ *is antitone then* $(L; \wedge)$ *is a lattice and for every* $x, y \in L$, $x \vee y = (x' \wedge y')'$.

We introduce the following notation: let $\mathscr{S} = (S; \wedge, 0)$ be a semilattice with the least element 0 such that for each $a \in S$ the interval $[0, a]$ is an ortholattice (with respect to the induced order). Denote by x^a the orthocomplement of $x \in [0, a]$ in this interval. We say that $\mathscr{S}$ satisfies the *compatibility condition* if for each $x \leqslant a \leqslant b$ of $\mathscr{S}$ the following equality is satisfied:

$$x^a = x^b \wedge a.$$

Definition. Let $\mathscr{S} = (S; \wedge, 0)$ be a semilattice with 0. $\mathscr{S}$ is called an *orthosemilattice* if the interval $[0, a]$ is an ortholattice for each $a \in S$. $\mathscr{S}$ is called an *orthomodular semilattice* if for each $a \in S$ the interval $[0, a]$ is an orthomodular lattice.

Clearly, every orthomodular semilattice is an orthosemilattice.

Proposition. *Every orthosemilattice satisfying the compatibility condition is an orthomodular semilattice.*

Proof. Let $\mathscr{S} = (S; \wedge, 0)$ be an orthosemilattice satisfying the compatibility condition. Let $a \in S$ and $x, y \in [0, a]$. Suppose $x \leqslant y$. Then $x \leqslant y \leqslant a$ and we have in the lattice $[0, a]$

$$x \vee (x^a \wedge y) = x \vee x^y = y.$$

Thus, every interval $[0, a]$ of $\mathscr{S}$ satisfies the orthomodular law. $\square$

Remark. The converse statement of Proposition does not hold in general, see the orthosemilattice $\mathscr{S}$ in Fig. 1, where we put $x^q = y$, $y^q = x$, $z^q = v$, $v^q = z$ and e.g. $x^1 = d$.

Evidently, $\mathscr{S}$ is an orthomodular semilattice but it does not satisfy the compatibility condition since $x \leqslant q \leqslant 1$ but $x^q = y \neq v = d \wedge q = x^1 \wedge q$.

Example. An orthosemilattice which is not an orthomodular semilattice is depicted in Fig. 2.

Now, we are ready to formulate our results.

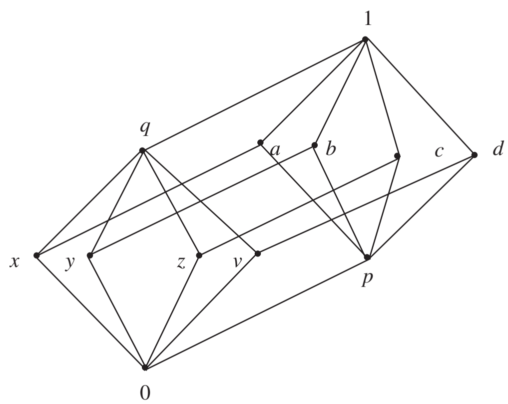

Fig. 1.

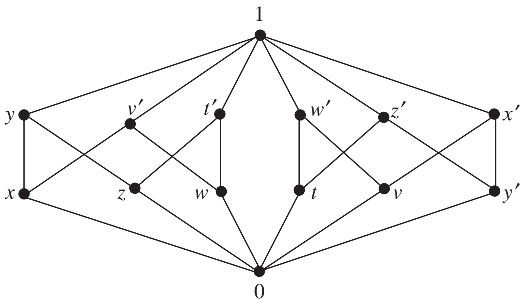

Fig. 2.

Theorem. *Let $\mathscr{S} = (S; \wedge, 0)$ be a semilattice with* 0. *The following conditions are equivalent*:

(1) *$\mathscr{S}$ is an orthosemilattice*,
(2) *there exists a binary operation $s(x, y)$ on S satisfying the identities*:
(A1) $s(x, y) \wedge (x \wedge y) = 0$,
(A2) $s(s(x, y), y) = x \wedge y$,
(A3) $s(x \wedge y, z) \wedge s(y, z) = s(y, z)$.

Proof. (1) ⇒ (2): Define $s(x, y) = (x \wedge y)^y$. Then clearly $x \wedge y \in [0, y]$ and hence also $(x \wedge y)^y \in [0, y]$. We can easily verify

(A1) $s(x, y) \wedge (x \wedge y) = (x \wedge y)^y \wedge (x \wedge y) = 0$,
(A2) $s(s(x, y), y) = ((x \wedge y)^y \wedge y)^y = (x \wedge y)^{yy} = x \wedge y$,
(A3) $s(x \wedge y, z) \wedge s(y, z) = (x \wedge y \wedge z)^z \wedge (y \wedge z)^z = (y \wedge z)^z = s(y, z)$ since $(x \wedge y \wedge z)^z \geqslant (y \wedge z)^z$.

(2) ⇒ (1): Let $\mathscr{S} = (S; \wedge, 0)$ be a semilattice with 0 and $s(x, y)$ be a binary operation on S satisfying (A1), (A2), (A3). Applying (A2) twice and using (A3), we get

$$s(x, y) \geqslant s(x, y) \wedge y = s(s(s(x, y), y), y) = s(x \wedge y, y) \geqslant s(x, y),$$

from which we infer

$$s(x, y) \wedge y = s(x, y).$$

For $a \in S$ and $x \in [0, a]$ we put $x^a = s(x, a)$. Clearly $x^a \in [0, a]$. By (A2) we conclude $(x^a)^a = x$ and, by (A3), $x \leqslant y \Rightarrow y^a \leqslant x^a$ for $x, y \in [0, a]$. Together, $x \mapsto x^a$ is an antitone involutory unary operation on $[0, a]$ and, by Lemma,

$$x \vee_a y = (x^a \wedge y^a)^a$$

is the supremum of x, y in $[0, a]$. Hence, $([0, a], \wedge, \vee_a)$ is a (bounded) lattice for each $a \in S$. By (A1),

$$x^a \wedge x = s(x, a) \wedge (x \wedge a) = 0$$

for $x \in [0, a]$ and, due to Lemma, x^a is a complement of x in $[0, a]$. Thus, $[0, a]$ is an ortholattice for each $a \in S$. □

For orthomodular semilattices, we can derive the following characterization:

Corollary 1. *Let $\mathscr{S} = (S; \wedge, 0)$ be a semilattice with* 0. *The following conditions are equivalent*:

(1) *$\mathscr{S}$ is an orthomodular semilattice*,
(2) *there exists a binary operation s on S satisfying the identities* (A1), (A2), (A3) *and* (A4)* $s(s(x \wedge y, z) \wedge y, z) \wedge y = x \wedge y \wedge z$.

Proof. By Theorem, we need only verify the equivalence of (A4)* with the orthomodular law for $s(x, y) = (x \wedge y)^y$.

Of course, the left-hand side of (A4)* can be rewritten as $((x \wedge y \wedge z)^z \wedge y \wedge z)^z \wedge y$. Since $((x \wedge y \wedge z)^z \wedge y \wedge z)^z \leqslant z$, we can write it as $((x \wedge y \wedge z)^z \wedge (y \wedge z))^z \wedge (y \wedge z)$. Hence, applying one of the De Morgan laws, (A4)* is

$$((x \wedge y \wedge z) \vee (y \wedge z)^z) \wedge (y \wedge z) = x \wedge y \wedge z$$

which is just the orthomodular law in $[0, z]$. We are done. □

Finally, we are able also to characterize orthomodular semilattices which satisfy the compatibility condition:

Corollary 2. *Let* $\mathscr{S} = (S; \wedge, 0)$ *be a semilattice with* 0. *The following conditions are equivalent*:

(1) $\mathscr{S}$ *is an orthomodular semilattice satisfying the compatibility condition*,
(2) *there exists a binary operation s on S satisfying the identities*: (A1), (A2), (A3) *and* (A4) $s(x \wedge y, z) \wedge y = s(x, y \wedge z)$.

Proof. By Theorem and Proposition, we need only verify the equivalence of (A4) with the compatibility condition for $s(x, y) = (x \wedge y)^y$. The left-hand side of (A4) is $(x \wedge y \wedge z)^z \wedge y = (x \wedge y \wedge z)^z \wedge y \wedge z$ since $(x \wedge y \wedge z)^z \leqslant z$ and the right-hand side is $(x \wedge y \wedge z)^{y \wedge z}$. The equivalence is now clear. □

Uncited reference

[2].

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